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2007 J. Phys. A: Math. Theor. 40 3729

(http://iopscience.iop.org/1751-8121/40/13/027)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 3729–3740 doi:10.1088/1751-8113/40/13/027

Modulational instability and exact solutions of the nonlinear Schrödinger equation coupled with the nonlinear Klein–Gordon equation

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Received 20 November 2006, in final form 9 February 2007
Published 14 March 2007
Online at stacks.iop.org/JPhysA/40/3729

Abstract

The modulational instability (MI) of the coupled nonlinear Schrödinger and nonlinear Klein–Gordon equations is investigated. It is found that there are a number of possibilities for the MI regions due to the generalized dispersion relation, which relates the frequency and wavenumber of the modulating perturbations. Some exact travelling wave solutions are constructed via the solutions of a ϕ^4 model through a simple mapping relation. Furthermore, we present five different types of solutions representing possible final states of modulationally unstable perturbations. The profiles of solitary wave structures are displayed for some fixed parameters.

PACS numbers: 52.35.−g, 05.45.Yv

1. Introduction

The nonlinear Schrödinger equation coupled with a nonlinear Klein–Gordon equation in the (1 + 1)-dimensional case reads

$$\mathrm{i}\frac{\partial\psi}{\partial t} + \alpha\frac{\partial^2}{\partial x^2}\psi + \rho\phi\psi = 0, \tag{1}$$

and

$$\frac{\partial^2\phi}{\partial t^2} - c_0^2\frac{\partial^2}{\partial x^2}\phi + \mu^2\phi + \gamma|\phi|^2\phi - \beta|\psi|^2 = 0. \tag{2}$$

The system of equations (1) and (2), which is known as the coupled Schrödinger–Klein–Gordon (S-KG) model, is a classical example describing a system of conserved scalar nucleons interacting with neutral scalar mesons where the dynamics of these fields are coupled through the Yukawa interaction [1]. In this context, ψ represents a complex scalar nucleon field and

ϕ a real scalar meson field, the real constant μ is the mass of a meson, g and ρ are quadratic coupling constants, the γ-term describes the auto-interaction, α and c_0^2 are constants.

Both the nonlinear Schrödinger (NLS) and nonlinear Klein–Gordon (KG) equations have been widely used to study the dynamics of small but finite amplitude nonlinearly interacting perturbations in many-body physics [2], in nonlinear optics [3] and optical communications [4], in nonlinear plasmas [5, 6] and complex geophysical flows [7], as well as in intense laser–plasma interactions and nonlinear quantum electrodynamics [8]. For example, the NLSE with a cubic nonlinearity is a suitable model for the nonlinear pulse propagation in Kerr media [10], photonics [9] and optical fibre communications [4, 11, 12], as well as in unmagnetized plasmas [13, 14, 18, 19], while the nonlinear KG equation [16] without a driver governs the dynamics of nonlinear trapped ion modes [15] in nonuniform magnetoplasmas. Furthermore, equations similar to (1) and (2) may describe the dynamics of coupled electrostatic upper-hybrid and ion-cyclotron waves in a uniform magnetoplasma [17].

One of the distinguishing features of the present system of equations (1) and (2) is the presence of a cubic auto-interaction effect in the Klein–Gordon field. Without the auto-interaction, namely $\gamma = 0$, the system (also in higher dimensions (2 + 1) or (3 + 1)) has been investigated, such as its Cauchy problem and initial boundary value problem [1, 20], the solitary wave solutions [21], etc. The existence and uniqueness of global solutions for rough data of the S-KG system with quadratic coupling and cubic auto-interactions have been proved recently in [22].

In this paper, we use equations (1) and (2) to investigate the modulational instabilities and their possible stationary states in the form of localized structures. The paper is organized in the following fashion. In section 2, we derive a nonlinear dispersion relation which depicts the modulational instabilities of a constant amplitude carrier wave. Possible exact travelling wave solutions of (1) and (2) are obtained in section 3. Specifically, five types of explicit analytical solutions are presented, and their relations to modulationally stable or unstable scenario are established. The profiles of different nonlinear excitations are graphically exhibited. A summary and discussion of our work is contained in section 4.

2. Modulational instability analysis

The modulational instability analysis is usually carried out by means of the following scheme. First, we have to find an equilibrium state of the system of equations under investigation, which is simple and exact monochromatic wave solutions. Second, we have to add a small perturbation on the equilibrium state with a perturbation wavenumber and frequency, which are much smaller than those of the carrier wave. The small perturbation functions satisfy a set of equations from which one deduces the nonlinear dispersion relation. The latter is analysed to obtain a complex frequency, revealing the growth of the amplitude-modulated wave packet. Based on the above idea, we analyse the coupled S-KG equations (1) and (2) in this section.

An equilibrium state can be obtained by inserting the assumption

$$\psi = \psi_0\, \mathrm{e}^{\mathrm{i}\omega t}, \qquad \phi = \phi_0, \tag{3}$$

where the constants ω, ϕ_0 are real and ψ_0 is complex, into the coupled S-KG equations (1) and (2). A simple calculation gives

$$\omega = \rho\phi_0, \tag{4}$$

and

$$\gamma\phi_0^3 + \mu^2\phi_0 - \beta|\psi_0|^2 = 0. \tag{5}$$

Next, we introduce a small perturbation around the above stationary state and linearize the coupled S-KG equations by substituting

$$\psi = (\psi_0 + \epsilon\psi_1)\,\mathrm{e}^{\mathrm{i}\rho\phi_0 t}, \qquad \phi = \phi_0 + \epsilon\phi_1, \tag{6}$$

with (5), into (1) and (2). Writing $\psi_1 = u + \mathrm{i}v$, $\psi_0 = a + \mathrm{i}b$ (u, v, a, b are real) and separating the real and imaginary parts of the linearized equations (the first-order terms of ϵ), we obtain

$$\alpha\frac{\partial^2 u}{\partial x^2} - \frac{\partial v}{\partial t} + a\rho\phi_1 = 0, \tag{7}$$

$$\alpha\frac{\partial^2 v}{\partial x^2} + \frac{\partial u}{\partial t} + b\rho\phi_1 = 0, \tag{8}$$

and

$$\frac{\partial^2\phi_1}{\partial t^2} - c_0^2\frac{\partial^2\phi_1}{\partial x^2} + \mu^2\phi_1 + 3\gamma^2\phi_0^2\phi_1 - 2\beta(au + bv) = 0. \tag{9}$$

Now we let u, v and ϕ_1 vary as $u_0\,\mathrm{e}^{\mathrm{i}(Kx-\Omega t)} + cc$, $v_0\,\mathrm{e}^{\mathrm{i}(Kx-\Omega t)} + cc$, $\phi_{10}\,\mathrm{e}^{\mathrm{i}(Kx-\Omega t)}$ + c.c., where K and Ω are the perturbation wavenumber and the frequency, respectively, and c.c. stands for the complex conjugate. After some straightforward calculations, we obtain the nonlinear dispersion relation

$$\Omega^4 - P\Omega^2 + Q = 0, \tag{10}$$

where

$$P = \alpha^2K^4 + c_0^2K^2 + \mu^2 + 3\gamma\phi_0^2, \tag{11}$$

and

$$Q = \alpha^2c_0^2K^6 + \alpha^2\big(\mu^2 + 3\gamma\phi_0^2\big)K^4 - 2\alpha\rho\phi_0\big(\mu^2 + \gamma\phi_0^2\big)K^2. \tag{12}$$

Obviously, equation (10) has the solution

$$\Omega_\pm^2 = \tfrac{1}{2}\big[P \pm \sqrt{P^2 - 4Q}\big]. \tag{13}$$

It is noted that the coupled S-KG equations are modulationally stable for any wavenumber K if and only if $\Omega_\pm^2$ are both positive real numbers. It is easy to check that, in order to have positive real $\Omega_\pm^2$, the following three conditions should be simultaneously satisfied:

$$P > 0, \qquad Q > 0, \qquad \Delta > 0, \tag{14}$$

where $\Delta = P^2 - 4Q$ is the discriminant quantity given by

$$\Delta = \big(\alpha^2K^4 - c_0^2K^2 - \mu^2 - 3\gamma\phi_0^2\big)^2 + 8\alpha\rho\beta|\psi_0|^2K^2. \tag{15}$$

The first stability condition $P > 0$ is easily satisfied for any K when $\mu^2 + 3\gamma\phi_0^2 > 0$. Otherwise, there is a critical value

$$K_{cr,1} = \left\{-\frac{c_0^2}{2\alpha^2} + \frac{1}{2}\left[\frac{c_0^4}{\alpha^4} - \frac{4\big(\mu^2 + 3\gamma\phi_0^2\big)}{\alpha^2}\right]^{1/2}\right\}^{1/2} \tag{16}$$

below which P is negative. In this case, we either have $\Omega_-^2 < 0 < \Omega_+^2$ or $\Omega_-^2 < \Omega_+^2 < 0$. It is remarkable that if there is no auto-interaction effect in the coupled S-KG equations, namely, $\gamma = 0$, then this first stability condition ($P > 0$) is always satisfied for any perturbation wavenumber K.

Let us consider the second stability condition $Q > 0$ in detail. We see that $Q = 0$ has two nonzero roots for K^2, namely,

$$K_{Q\pm}^2 = -\frac{\mu^2 + 3\gamma\phi_0^2}{2c_0^2} \pm \frac{1}{2}\left[\left(\frac{\mu^2 + 3\gamma\phi_0^2}{c_0^2}\right)^2 + \frac{8\beta\rho|\psi_0|^2}{\alpha c_0^2}\right]^{1/2}$$
$$\equiv -\frac{\mu^2 + 3\gamma\phi_0^2}{2c_0^2} \pm \frac{1}{2}\sqrt{\Delta_Q}, \qquad (17)$$

with (5). Therefore, $Q > 0$ for any K requires either

(i) that $\Delta_Q < 0$. This is only possible for the perturbation amplitudes ψ_0 and ϕ_0 satisfying a specific relation besides (5). Thus, this case cannot be generally ensured for arbitrary perturbation amplitudes or

(ii) that $\Delta_Q > 0$ and $K_{Q\pm}^2$ are both negative real values. It is ensured if $\mu^2 + 3\gamma\phi_0^2 > 0$ and $\alpha\beta\rho < 0$. Otherwise, if $\mu^2 + 3\gamma\phi_0^2 < 0$ and $\alpha\beta\rho < 0$, then we have $K_{Q+}^2 > K_{Q-}^2 > 0$ and it will be unstable in the region $K_{Q-}^2 < K^2 < K_{Q+}^2$. If $\alpha\beta\rho > 0$, then $K_{Q-}^2 < 0 < K_{Q+}^2$ and it is unstable when $K^2 < K_{Q+}^2$.

Finally, let us check the last stability condition $\Delta > 0$. Evidently, this condition is always satisfied when $Q < 0$. However, for $Q > 0$, this condition is ensured for any K when $\alpha\beta\gamma > 0$. Otherwise, we have to consider the inequality

$$\Delta = d_8K^8 + d_6K^6 + d_4K^4 + d_2K^2 + d_0 > 0, \qquad (18)$$

with

$$d_8 = \alpha^4, \qquad d_6 = -2c_0^2\alpha^2, \qquad d_4 = c_0^4 - 2\alpha^2(\mu^2 + 3\gamma\phi_0^2),$$
$$d_2 = 2c_0^2(\mu^2 + 3\gamma\phi_0^2) + 8\alpha\rho\beta|\psi_0|^2, \qquad d_0 = (\mu^2 + 3\gamma\phi_0^2)^2. \qquad (19)$$

In order to identify the region for K where the waves are unstable, we have to find the roots for $\Delta = 0$. It is not so easy as the previous conditions, since we have to employ the existing complicated analytical formulae and the associated criteria for the roots of an eighth-order polynomial (or a fourth order for K^2).

We note that the S-KG system is modulationally unstable when one or more of the above conditions are violated. For example, the growth rate of instability is $\sigma = \sqrt{-\Omega_-^2}$ for $P < 0$, $Q < 0$, and the corresponding wavenumber ranges are determined by $[0, K_{\mathrm{cr},1}]$ and either $[0, K_{Q+}]$ or $[K_{Q-}, K_{Q+}]$, depending on the parameter values. The instability for this case is manifested as a purely growing mode. For $\Delta < 0$, all the solutions of (10) are complex. Consequently, the growth rate σ is determined by the imaginary part $\mathrm{Im}(\Omega_\pm^2) = \pm\sqrt{|\Delta|}/2$.

From the above analysis, we note that several different unstable wavenumber regimes may appear, either partially superimposed or distinct from each other. It is shown that instability growth rate may be dramatically affected by the auto-interaction. When the system has a negative auto-interaction, then it may have a higher instability growth rate with an enlarged unstable wavenumber region, as indicated in figure 1.

3. Exact solutions

Many exact solutions of the coupled S-KG equations (1) and (2) with $\gamma = 0$ have been obtained [21]. In the following, we present stationary solutions of equations (1) and (2).

In order to obtain some exact stationary solutions, we first introduce the assumption $\psi = u\exp(\mathrm{i}kx - \mathrm{i}\omega t)$, where $u(x, t)$ is a real function and k and ω are constants. With this

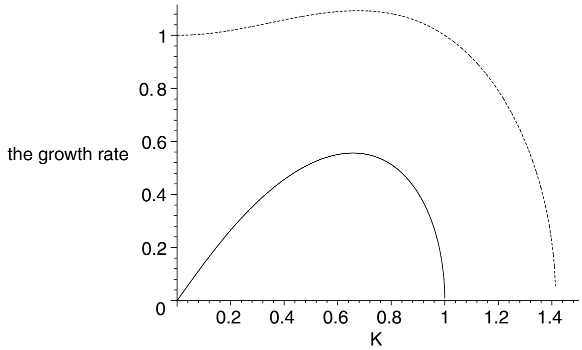

Figure 1. The instability growth rate $\sigma = \sqrt{-\Omega_-^2}$ is plotted versus the perturbation wavenumber K for the parameter values $\alpha = c_0 = \rho\beta = |\psi_0|^2 = 1$ and $\mu^2 + 3\phi_0^2\gamma = 1$ for the solid line and $\mu^2 + 3\phi_0^2\gamma = -1$ for the dotted line.

ansatz, we obtain from (1) and (2)

$$\alpha \frac{\partial^2 u}{\partial x^2} + (\omega - \alpha k^2 + \rho\phi)u = 0, \tag{20}$$

$$\frac{\partial u}{\partial t} + 2\alpha k \frac{\partial u}{\partial x} = 0, \tag{21}$$

and

$$\frac{\partial^2 \phi}{\partial t^2} - c_0^2 \frac{\partial^2 \phi}{\partial x^2} + \mu^2\phi + \gamma\phi^3 - \beta u^2 = 0. \tag{22}$$

The general solution of (21) reads

$$u = u(a(x - 2\alpha k t)) \equiv u(X), \tag{23}$$

so that equation (20) becomes

$$\alpha a^2 \frac{\partial^2 u}{\partial X^2} + (\omega - k^2\alpha)u + \rho\phi u = 0. \tag{24}$$

Along the travelling line $X = a(x - 2\alpha k t)$, equation (22) can be rebuilt by $\phi(x, t) \equiv \phi(X)$ as

$$a^2\left(c_0^2 - 4\alpha^2 k^2\right)\frac{\partial^2 \phi}{\partial X^2} + \beta u^2 - \mu^2\phi - \gamma\phi^3 = 0. \tag{25}$$

Now, we are left to solve equations (24) and (25). They cannot, however, be solved simply by integration due to the coupling between u and ϕ.

In the past, a deformation mapping method [25] (and references therein) has been used to solve nonlinear differential equations. Furthermore, the F-expansion method and its generalization have been widely used to solve the coupled S-KG equations with $\gamma = 0$ [21].

In the following, we will simply deform the solutions of equations (24) and (25) to the solutions of the ϕ^4 model

$$f_X^2 = Pf^4 + Qf^2 + R, \tag{26}$$

with constants P, Q and R. A large number of exact solutions of (26) can be found in [21, 25], which can then be used to construct the solutions of the coupled S-KG equations.

A type of mapping relation can be easily established by using the idea of the truncated Painlevé approach [26]. A straightforward calculation gives a simple polynomial expansion type relation

$$u = U_0 + U_1 f + U_2 f^2 + U_3 f^3, \qquad \phi = V_0 + V_1 f + V_2 f^2, \tag{27}$$

where $U_i, V_j (i = 0, 1, 2, 3, j = 0, 1, 2)$ are constants, and f is given by (26).

Substituting (27) into (24) and (25), and taking advantage of (26) to cancel all the derivatives of f_{nX} $(n > 1)$ and the powers of f_X^n $(n > 1)$, collecting all the coefficients of different powers of f (the terms with f_X have been cancelled), we obtain a set of algebraic equations

$$(12a^2\alpha P + \rho V_4)U_6 = 0, \qquad (6\alpha a^2 P + \rho V_4)U_5 + \rho V_3 U_6 = 0, \qquad \gamma V_4^3 - \beta U_6^2 = 0, \tag{28}$$

$$2\alpha a^2 U_5 R + (\rho V_2 - k^2\alpha + \omega)U_3 = 0, \quad 3\gamma V_3 V_4^2 - 2\beta U_5 U_6 = 0, \tag{29}$$

$$-2\beta U_4 U_5 - 2\beta U_3 U_6 + 6\gamma V_2 V_3 V_4 + \gamma V_3^3 + 2a^2\left(4\alpha^2 k^2 - c_0^2\right) P V_3 = 0, \tag{30}$$

$$3\gamma V_3^2 V_4 + 6a^2\left(4\alpha^2 k^2 - c_0^2\right) P V_4 - \beta U_5^2 + 3\gamma V_2 V_4^2 - 2\beta U_4 U_6 = 0, \tag{31}$$

$$(\omega - k^2\alpha + \rho V_2 + 4\alpha a^2 Q)U_5 + \rho V_3 U_4 + \rho U_3 V_4 = 0, \tag{32}$$

$$(\omega - \alpha k^2 + \rho V_2 + \alpha a^2 Q)U_4 + 6\alpha a^2 R U_6 + \rho U_3 V_3 = 0, \tag{33}$$

$$(\omega - \alpha k^2 + \rho V_2 + 9\alpha a^2 Q)U_6 + \rho V_3 U_5 + (2\alpha a^2 P + \rho V_4)U_4 = 0, \tag{34}$$

$$4a^2\left(4\alpha^2 k^2 - c_0^2\right) Q V_4 - 2\beta U_3 U_5 + \mu^2 V_4 - \beta U_4^2 + 3\gamma V_2\left(V_2 V_4 + V_3^2\right) = 0, \tag{35}$$

$$\mu^2 V_3 - 2\beta U_3 U_4 + a^2\left(4\alpha^2 k^2 - c_0^2\right) Q V_3 + 3\gamma V_2^2 V_3 = 0, \tag{36}$$

$$\mu^2 V_2 - \beta U_3^2 + 2a^2\left(4\alpha^2 k^2 - c_0^2\right) R V_4 + \gamma V_2^3 = 0. \tag{37}$$

Solving equations (28)–(37) by *Maple*, we obtain

$$U_0 = U_2 = V_1 = 0, \qquad V_0 = -\frac{\rho\left(c_0^2 - 4\alpha^2 k^2\right)}{6\gamma\alpha} - \frac{8\alpha a^2 P U_1}{\rho U_3}, \tag{38}$$

$$V_2 = -\frac{12\alpha a^2 P}{\rho}, \qquad \omega = \alpha k^2 + \frac{\rho^2\left(c_0^2 - 4\alpha^2 k^2\right)}{24\alpha\gamma} - \frac{\alpha^2\gamma\beta U_1^3}{\rho\left(c_0^2 - 4\alpha^2 k^2\right)^2 U_3}, \tag{39}$$

$$U_3^2 = -\frac{1728\gamma\alpha^3 a^6 P^3}{\rho^3\beta}, \qquad Q = \frac{\rho^2\left(c_0^2 - 4\alpha^2 k^2\right)}{72\gamma\alpha^2 a^2} + \frac{2PU_1}{U_3} + \frac{\alpha\gamma\beta U_1^3}{9a^2\rho\left(c_0^2 - 4\alpha^2 k^2\right)^2 U_3}, \tag{40}$$

$$R = \frac{\rho^2\left(c_0^2 - 4\alpha^2 k^2\right) U_1}{54\gamma\alpha^2 a^2 U_3} + \frac{PU_1^2}{U_3^2} + \frac{4\gamma\beta\alpha U_1^4}{27a^2\rho\left(c_0^2 - 4\alpha^2 k^2\right)^2 U_3^2}, \tag{41}$$

$$\mu^2 = -\frac{\rho^2\left(c_0^2 - 4\alpha^2 k^2\right)^2}{36\gamma\alpha^2} + \frac{\beta\rho U_1^2}{36\alpha a^2 P} + \frac{4\alpha\gamma\beta U_1^3}{9\rho\left(c_0^2 - 4\alpha^2 k^2\right) U_3}. \tag{42}$$

Hence, the travelling wave solutions of equations (1) and (2) are

$$\psi = (U_1 + U_3 f^2) f \exp\left\{ \mathrm{i}\left[kx - \left(\alpha k^2 + \frac{\rho^2\left(c_0^2 - 4\alpha^2 k^2\right)}{24\alpha\gamma} - \frac{\alpha^2\gamma\beta U_1^3}{\rho\left(c_0^2 - 4\alpha^2 k^2\right)^2 U_3} \right) t \right] \right\}, \tag{43}$$

and

$$\phi = -\frac{\rho\left(c_0^2 - 4\alpha^2 k^2\right)}{6\gamma\alpha} - \frac{8\alpha a^2 P U_1}{\rho U_3} - \frac{12\alpha a^2 P}{\rho} f^2, \tag{44}$$

where $f \equiv f(X)$ satisfies (26), $X = a(x - 2\alpha k t)$, U_1, U_3, a and k are determined by (40)–(42).

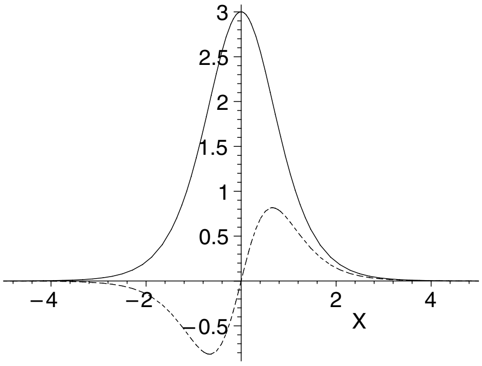

Figure 2. Plot of the amplitude of ψ with the upper sign (dotted line) and ϕ (solid line) expressed by (45) and (46), respectively, with the parameter values $\mu = \alpha = \rho = \beta = 1, \gamma = -\frac{1}{6}$.

In the following, we list some simple explicit solutions. For simplicity, if we choose $P = 1, Q = -2, R = 1$, then the solution of (26) is $f = \tanh(X)$. In this case, we obtain two types of solutions.

Solution 1

$$\psi = \pm 3\mu^2 \sqrt{\frac{\alpha}{2\rho\beta}\sqrt{-\frac{\rho^2}{6\gamma\mu^2\alpha^2}}} \tanh(X)\,\mathrm{sech}^2(X) \exp\left(\mathrm{i}\left(kx - \frac{\alpha\left(\mu^2 + 4k^4\alpha^2 - k^2c_0^2\right)}{4\alpha^2k^2 - c_0^2}t\right)\right), \tag{45}$$

and

$$\phi = \frac{3\mu^2\alpha}{\rho}\sqrt{-\frac{\rho^2}{6\gamma\mu^2\alpha^2}}\,\mathrm{sech}^2(X), \tag{46}$$

with

$$k = \pm\frac{1}{2}\sqrt{\frac{c_0^2}{\alpha^2} - \sqrt{-\frac{6\gamma\mu^2}{\rho^2\alpha^2}}}, \qquad a = \pm\sqrt{\frac{\mu^2}{4}\sqrt{-\frac{6\gamma\mu^2\alpha^2}{\rho^2}}}. \tag{47}$$

Evidently, for this type of solution, we must have $\gamma < 0$ and $\alpha\beta\rho > 0$; therefore, from the modulationally instability analysis in the last section, it can be deduced that this solution is associated with modulationally unstable perturbations in some regions due to $Q < 0$ (or $P < 0$ too).

The structure of this solution is plotted in figure 2 for some parameter values. It shows that the two waves have vanishing boundary conditions at infinity, namely, $\phi, \psi \to 0$ when $x \to \pm\infty$. The wave governed by the KG field has a larger positive amplitude, while the other wave is weak and asymmetric. In this case, the resultant field in the near positive x region might be strengthened and the near negative region might be weakened, which means that these two waves might interact in a complicated fashion in the region around the centre. In other words, the phenomenon described in figure 1 can be viewed as an interaction between a bright soliton and a dipole-type soliton.

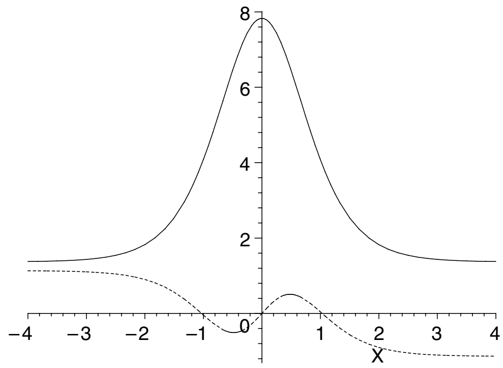

Figure 3. Plot of the amplitude of ψ with the upper sign (dotted line) and ϕ (solid line) expressed by (48) and (49), respectively, where the parameter values are $c_0 = \beta = \alpha = \rho = \mu = 1$, $\gamma = -0.03$, a and k are determined as $\sim$0.73 and $\sim$0.12, respectively.

Solution 2

$$\psi = \pm\frac{24}{5}\sqrt{-\frac{3\gamma a^6\alpha^3}{\rho^3\beta}}[3 - 5\tanh^2(X)]\tanh(X) \times \exp\left(\mathrm{i}\left(kx - \frac{6\gamma\alpha^2(36a^2 + 5k^2) - 5\rho^2\left(4\alpha^2k^2 - c_0^2\right)}{30\gamma\alpha}t\right)\right), \tag{48}$$

and

$$\phi = \frac{\rho\left(4\alpha^2k^2 - c_0^2\right)}{6\alpha\gamma} + \frac{12a^2\alpha}{5\rho}[2 - 5\tanh^2(X)], \tag{49}$$

with k and a determined by

$$125\rho^6\left(4\alpha^2k^2 - c_0^2\right)^3 - 7200\rho^4\alpha^2\gamma\left(4\alpha^2k^2 - c_0^2\right)^2a^2 - 373\,248\gamma^3a^6\alpha^6 = 0, \tag{50}$$

$$25\rho^4\left(4\alpha^2k^2 - c_0^2\right)^2 - 960\gamma\alpha^2a^2\left(4\alpha^2k^2 - c_0^2\right)\rho^2 + 12\alpha^2\gamma(432\gamma\alpha^2a^4 + 25\mu^2\rho^2) = 0. \tag{51}$$

In this class of solution, it is required that $\gamma\alpha\rho\beta < 0$. In order to have real solutions of a and k from (50) and (51), after some trial and error, we find that γ might be a small negative value (not proven). Therefore, similar to solution 1, it will be associated with modulationally unstable perturbations in some regions for $Q < 0$ (or $P < 0$ too).

For some parameter values, this solution can have structures, as shown in figure 3. It is seen that the wave governed by the KG field approaches a positive value when $x \to \pm\infty$, while the other wave has a same asymptotic value when $x \to -\infty$ and a different negative value when $x \to \infty$. The wave governed by the NLS equation can be viewed as a kind of shock wave with a small oscillation around the centre. In this case, the coupled S-KG system can support two different types of waves, say, a bright solitary wave and a shock wave. Similar to the waves in figure 2, the wave governed by the KG field plays a dominant role.

If we choose a solution of (26) as $f = \mathrm{sech}(X)$ for $P = -1$, $Q = 1$, $R = 0$, we then obtain three types of explicit solutions.

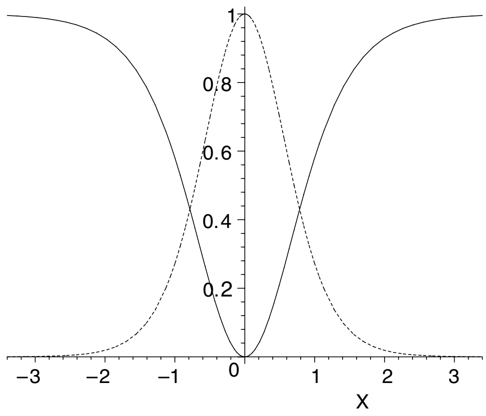

Figure 4. Plot of the amplitude of ψ with the upper sign (dotted line) and ϕ (solid line) expressed by (52) and (53), respectively, with the parameter values $\mu = \beta = 1$, $\gamma = -1$.

Solution 3

$$\psi = \pm\frac{1}{16}\sqrt{-\frac{30\mu^3 A}{\beta}}[4 - 5\,\mathrm{sech}^2(X)]\,\mathrm{sech}(X) \times \exp\left(\mathrm{i}\left(kx - \frac{8\rho A c_0^2\gamma - 48\mu\alpha\gamma + 5\mu\rho^2\alpha}{32\gamma\rho A\alpha}t\right)\right), \tag{52}$$

and

$$\phi = \frac{5}{4}\mu A\,\mathrm{sech}(X)^2, \tag{53}$$

where

$$A = \pm\sqrt{-\frac{3}{2\gamma}}$$

and

$$k = \pm\sqrt{\frac{c_0^2}{4\alpha^2} - \frac{3\mu}{2\rho A\alpha}}, \qquad a = \pm\frac{1}{4}\sqrt{\frac{5\rho A\mu}{3\alpha}}. \tag{54}$$

It is obvious that $\gamma < 0$. After some simple deduction, we find that $\alpha\rho\beta < 0$. Consequently, this class of solution may also be associated with modulationally unstable perturbations due to $Q < 0$ (or $P < 0$ too).

Different from the previous two figures, figure 4 shows two waves in similar structures. The wave governed by the KG field has a nonzero (positive) boundary value, while the other wave has a zero boundary value when $x \to \pm\infty$. Hence, we can say that the KG field can support a dark soliton while the NLS field supports a bright soliton, and thus the interaction phenomenon between a dark soliton and a bright soliton might be observed in the S-KG system.

Solution 4

$$\psi = \pm\sqrt{-\frac{\mu^3}{\gamma\beta A}}\,\mathrm{sech}^3(X)\exp\left(\mathrm{i}\left(kx - \frac{\rho c_0^2\gamma A - 6\gamma\mu\alpha + \mu\rho^2\alpha}{4\alpha\rho\gamma A}t\right)\right), \tag{55}$$

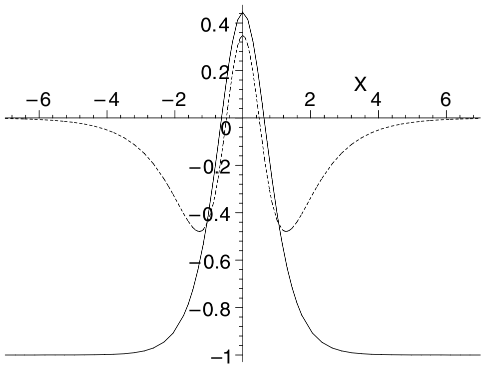

Figure 5. Plot of the amplitude of ψ with the upper sign (dotted line) and ϕ (solid line) expressed by (55) and (56), respectively, with the parameter values $\mu = \beta = 1$, $\gamma = -1$.

and

$$\phi = \mu A \tanh^2(X), \tag{56}$$

where

$$A = \pm\sqrt{-\frac{1}{\gamma}}$$

and

$$k = \pm\sqrt{\frac{c_0^2}{4\alpha^2} + \frac{3\gamma A\mu}{2\rho\alpha}}, \qquad a = \pm\frac{1}{2}\sqrt{\frac{\mu\rho}{3\alpha\gamma A}}. \tag{57}$$

Similar to the solution 3, for real A, we require $\gamma < 0$, and no matter what sign one chooses for A, we must have $\alpha\rho\beta < 0$. Therefore, this class of solution is also associated with modulationally unstable perturbations for the same reason as the previous solutions.

Figure 5 is plotted for this solution. Contrary to the waves shown in figure 4, now the wave governed by the KG field has a zero boundary value, while the other wave has a nonzero (negative) boundary value when $x \to \pm\infty$. Again, the two waves have totally different structures around the centre. In this case, the KG field supports a grey soliton, while the NLS field supports a W-type soliton. Hence, the interaction between a grey soliton and a W-type soliton might be observed in the system with some particular parameters.

Solution 5

$$\psi = \pm\frac{24}{5}\sqrt{\frac{3\gamma\alpha^3 a^6}{\rho^3\beta}}[4 - 5\,\mathrm{sech}^2(X)]\,\mathrm{sech}(X)\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}, \tag{58}$$

and

$$\phi = \frac{25\mu^2\rho^2 - 768\gamma a^4\alpha^2}{300\gamma\rho\alpha a^2} + \frac{12\alpha a^2}{\rho}\,\mathrm{sech}^2(X), \tag{59}$$

where

$$\begin{aligned} k &= \pm\sqrt{\frac{144\gamma a^2}{25\rho^2} + \frac{c_0^2}{4\alpha^2} + \frac{\mu^2}{8\alpha^2 a^2}}, \\ \omega &= \frac{3\alpha(48\gamma + 13\rho^2)a^2}{25\rho^2} + \frac{c_0^2}{4\alpha} - \frac{\mu^2(2\rho^2 - 3\gamma)}{24\alpha\gamma a^2}, \end{aligned} \tag{60}$$

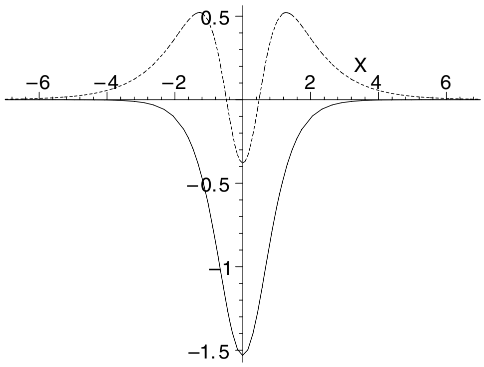

Figure 6. Plot of the amplitude of ψ with the upper sign (dotted line) and ϕ (solid line) expressed by (58) and (59), respectively, where the parameter values are $\mu = \rho = \alpha = 1$, $\beta = \gamma = -1$ and a is determined as $\frac{\sqrt{450+30\sqrt{33}}}{48} \sim 0.52$.

and a is determined by

$$589\,824\gamma^2\alpha^4a^8 + 51\,600\mu^2\rho^2\gamma\alpha^2a^4 + 625\mu^4\rho^4 = 0. \tag{61}$$

It is obvious that $\alpha\beta\gamma\rho > 0$. In order to obtain a real solution of a from equation (61), it can easily be proven that γ must be negative. Hence, $\alpha\beta\rho < 0$; which means that this type of solution is also associated with modulationally unstable perturbations because $Q < 0$ (or $P < 0$ too).

This type of solution is plotted in figure 6 for some parameter values. It is found that the two wave packages approach zero when $x \to \pm\infty$, and they also possess different structures around the centre. In this figure, a dark soliton is generated for the KG field and an M-type soliton for the NLS field, which means that an interesting interaction between a dark soliton and an M-type soliton might occur in the S-KG system under some particular parameters.

4. Summary and discussion

In this paper, we have studied the properties of the nonlinear Schrödinger equation coupled with the nonlinear Klein–Gordon equation that includes the cubic auto-interaction. Specifically, we have investigated the modulational instabilities and associated stationary nonlinear states of finite amplitude excitations that are governed by the S-KG equations (1) and (2). Standard techniques of the parametric instabilities have been used to derive the nonlinear dispersion relation (NDR) from the S-KG equations. The NDR reveals pronounced modulational instability in the presence of a negative cubic auto-interaction in the KG equation. Furthermore, we have found extended regimes for the modulationally unstable wavenumbers in our S-KG system of equations.

We have presented five different types of exact stationary solutions of equations (1) and (2). These solutions have been analytically constructed by exploiting the ϕ^4 model. For a set of parameters, we have worked out the profiles of the nonlinear structures numerically.

Figures 2–6 show numerically that the Klein–Gordon field shares a similar shape for all these solutions, while the amplitude of the function ψ assumes quite different shapes, which means that the S-KG system can have various nonlinear waves, such as the bright, dark, grey, dipole-type, W-type and M-type solitons and shock waves. Moreover, interesting interactions between a bright soliton and a dark or dipole-type or W-type solitons or a shock wave, and between a dark and an M-type soliton, might be observed in the S-KG system for different parameter values.

Our results for the modulational instabilities and associated nonlinear structure should be useful for understanding the propagation of nonlinearly interacting fields in magnetized plasmas, in nonlinear optics and in particle physics. Furthermore, it should be stressed that the dynamics of different classes of localized nonlinear solutions can be studied by solving equations (1) and (2) numerically, by using our stationary nonlinear solutions as initial conditions. However, such an investigation is beyond the scope of the present paper.

Acknowledgments

The authors thank B Eliasson for valuable discussions. XYT acknowledges the financial support from the Alexander von Humboldt Foundation and the support by the Youth Foundation of Shanghai Jiao Tong University and the National Natural Science Foundations of China (Nos. 10475055 and 10547124).

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